

Description of deformed nuclei in the sdg boson model

S.C. Li¹ and S. Kuyucak²

Department of Theoretical Physics,
Research School of Physical Sciences,
Australian National University, Canberra, ACT 0200, Australia

Abstract

We present a study of deformed nuclei in the framework of the sdg interacting boson model utilizing both numerical diagonalization and analytical $1/N$ expansion techniques. The focus is on description of high-spin states which have recently become computationally accessible through the use of computer algebra in the $1/N$ expansion formalism. A systematic study is made of high-spin states in rare-earth and actinide nuclei.

PACS: 21.60.Fw

Keywords: Interacting boson model, $1/N$ expansion, high-spin states.

¹ E-mail: scl105@rsphy1.anu.edu.au

² E-mail: sek105@phys.anu.edu.au

1 Introduction

Application of the interacting boson model (IBM) [1] to deformed nuclei has been fraught with conceptual and technical difficulties from the early days. Truncation of the model space to s and d bosons created a lot of controversy and was challenged on both microscopic [2] and phenomenological [3] grounds. When the dust finally settled, it was generally accepted that g bosons were both necessary and sufficient for microscopic foundation [4] as well as phenomenological description of deformed nuclei in the IBM [5] (see also [6, 7, 8] for recent reviews). The sdg-IBM, however, does not share many of the simplifying features of the sd-IBM which made it so popular. To start with, the symmetry limits of the dynamical group $SU(15)$ are not associated with actual spectra so there is little help from group theory. Secondly, due to the large basis spaces, exact diagonalization of sdg-IBM Hamiltonians is not possible for deformed nuclei. As a result, progress in sdg-IBM calculations has been rather slow, and due to various approximations involved, a satisfactory description of both low-lying band structures and high-spin states in deformed nuclei is still missing.

Numerical calculations in the sdg-IBM were initially performed by coupling a single g boson to an sd boson core [9]. In another approach, a Hamiltonian consisting of various $SU(3)$ tensor operators was diagonalized in a truncated $SU(3)$ basis [5]. At present, full basis sdg-IBM calculations are possible only for vibrational and transitional nuclei with boson numbers $N \leq 10$ [10]. The accuracy of the truncated space calculations for deformed nuclei with $N \geq 12$ has not been explored so far. One of the purposes of this paper is to examine the convergence properties of various matrix elements (m.e.) with the number of g bosons allowed, and to establish some criteria for the accuracy of truncated sdg-IBM calculations. It should be clear that due to truncation, the validity of numerical diagonalization results in the sdg-IBM is limited to low-lying states. Besides, such calculations are time consuming and therefore are not well suited to explore the effect of the multitude of parameters in the sdg-IBM. The $1/N$ expansion method [11], which is based on angular momentum projected mean field theory, could alleviate both problems. The available analytic formulas for various physical quantities allow systematic study of model parame-

ters, and fast and efficient analysis of data [12]. In addition, the $1/N$ expansion has recently been extended to higher orders using computer algebra [13], thus enabling accurate descriptions of high-spin states within the sdg-IBM. Currently the $1/N$ expansion formalism covers ground- and single-phonon bands. Calculations for multi-phonon bands are rather complicated and have not been performed yet. Therefore, at this stage, one has to rely on both analytical and numerical methods to obtain a complete description of deformed nuclei

In this paper, we first discuss the recent developments in the numerical diagonalization and the $1/N$ expansion methods. We then present a systematic study of deformed nuclei in the sdg-IBM employing a Hamiltonian consisting of one-body energies and quadrupole and hexadecapole interactions. Application of the results is focused on the high-spin states in rare-earth and actinide nuclei as their description in the IBM has been a source of criticism [3] which has not been adequately addressed so far. In order to constrain the model parameters properly, both the high-spin data and the low-lying band structures are described simultaneously.

2 Choice of Hamiltonian

The consistent- Q formalism (CQF) [6] has been the standard choice for description of deformed nuclei in the sd-IBM. The CQF Hamiltonian consists of the dipole and quadrupole interactions

$$H = \kappa_1 L \cdot L - \kappa_2 Q \cdot Q, \quad (1)$$

where the quadrupole operator is given by

$$Q = [s^\dagger \tilde{d} + d^\dagger \tilde{s}]^{(2)} + \chi [d^\dagger \tilde{d}]^{(2)}. \quad (2)$$

Here brackets denote tensor coupling of the boson operators and $\tilde{b}_{l\mu} = (-1)^\mu b_{l-\mu}$. For consistency, the same quadrupole operator is used in the E2 transition operator, $T(E2) = e_2 Q$, as in the Hamiltonian. The CQF has been especially successful in reproducing the low-lying band structures and the E2 transitions among them [6]. A more complete description of deformed nuclei, including high-spin states and hexadecapole bands, however, requires extension of the model space to sdg bosons.

A minimal extension of the CQF Hamiltonian to the sdg-IBM can be achieved by including the g-boson energy term, $\varepsilon_g \hat{n}_g$ in the Hamiltonian (1), and modifying the quadrupole operator (2) to

$$Q = [s^\dagger \tilde{d} + d^\dagger \tilde{s}]^{(2)} + q_{22}[d^\dagger \tilde{d}]^{(2)} + q_{24}[d^\dagger \tilde{g} + g^\dagger \tilde{d}]^{(2)} + q_{44}[g^\dagger \tilde{g}]^{(2)}. \quad (3)$$

We shall refer to this minimal extension as the CQF below. A study of high-spin states in the sdg-IBM using the CQF Hamiltonian has indicated that the energy surface remains too rigid, and inclusion of the d-boson energy term, $\varepsilon_d \hat{n}_d$, is essential in order to reproduce the spin dependence of moment of inertia (MOI) and E2 transitions in the ground band [14]. The success of this pairing plus quadrupole type of Hamiltonian, however, does not extend to the side bands which are more sensitive to interference from the hexadecapole interaction. Thus, for a comprehensive description of deformed nuclei one needs to employ the Hamiltonian

$$H = \varepsilon_d \hat{n}_d + \varepsilon_g \hat{n}_g - \kappa_2 Q \cdot Q - \kappa_4 T_4 \cdot T_4, \quad (4)$$

where the hexadecapole operator is given by

$$T_4 = [s^\dagger \tilde{g} + g^\dagger \tilde{s}]^{(4)} + h_{22}[d^\dagger \tilde{d}]^{(4)} + h_{24}[d^\dagger \tilde{g} + g^\dagger \tilde{d}]^{(4)} + h_{44}[g^\dagger \tilde{g}]^{(4)}. \quad (5)$$

Note that we have deliberately left out the dipole interaction in eq. (4) as it is at the root of the rigid MOI problem in the IBM. This Hamiltonian contains 10 parameters, namely, the one-body energies ε_d and ε_g , the multipole interaction strengths κ_2 and κ_4 , and the quadrupole and hexadecapole parameters q_{jl} and h_{jl} . In a systematic study covering many nuclei, it is desirable to have a smaller set of free parameters. To achieve this goal, we adapt a similar strategy as in a previous study of deformed nuclei [12]. The quadrupole parameters $\{q_{22}, q_{24}, q_{44}\}$ are scaled from their SU(3) values with a single factor q as suggested by microscopics [15], while the hexadecapole parameters h_{jl} are determined from those of q_{jl} through the commutation condition, $[\bar{h}, \bar{q}] = 0$, which ensures that the quadrupole and the hexadecapole mean fields are coherent

$$\bar{h}_{22} = \bar{q}_{24}, \quad \bar{h}_{24} = \bar{q}_{44}, \quad \bar{h}_{44} = \bar{q}_{24} + (\bar{q}_{44}^2 - \bar{q}_{22}\bar{q}_{44} - 1)/\bar{q}_{24}. \quad (6)$$

Here $\bar{q}_{jl} = \langle j0l0|20 \rangle q_{jl}$ and $\bar{h}_{jl} = \langle j0l0|40 \rangle h_{jl}$. This reduction of parameters from 10 to 5 is obtained at the expense of detailed description of quadrupole and hex-

adecapole operators. Since information, especially on the latter, is rather patchy, this will not cause any problems except in a few isolated cases. In the calculation of E2 and E4 transitions, we shall use the consistent operators, $T(\text{E2}) = e_2 Q$, $T(\text{E4}) = e_4 T_4$, so that, apart from the effective charges e_2 and e_4 , no new parameters are introduced.

In concluding this section, we present an alternative parametrization of the Hamiltonian (4) which is more convenient in implementation of the $1/N$ expansion formulas. This involves factoring out the energy scale and the leading order N dependence from the energy expressions. Since the quadrupole interaction is dominant, a suitable choice for such a set of dimensionless parameters is given by

$$\eta_l = \varepsilon_l / N \kappa_2, \quad \zeta_k = \kappa_k / \kappa_2, \quad (7)$$

where $l = 0, 2, 4, \dots$ correspond to the subscripts s, d, g, \dots

3 Numerical diagonalization

A computer code which can diagonalize arbitrary sdg-IBM Hamiltonians in full space has been available for some time [16], but due to excessive memory requirements it had only limited applications to transitional nuclei [10]. Recently this code has been modified to run on supercomputers improving its applicability [17]. Nevertheless, exact diagonalization for deformed nuclei with $N > 10$ remains elusive due to the large basis space, and truncation of the model space is still necessary. Here we use the modified code to study the accuracy of the truncated space calculations in the sdg-IBM. Numerical results will also be used in demonstrating the accuracy of the $1/N$ expansion formulas in sect. 4, and in application to actual spectra in sect. 6.

Since g bosons are relatively weakly coupled, a natural parameter in truncating the basis space is the maximum number of g bosons allowed, $n_{g\text{max}}$. In figs. 1-2, we examine the convergence properties of some key observables as a function of $n_{g\text{max}}$. For this purpose we use a typical deformed Hamiltonian with parameters (as defined in eq. (7)), $\kappa_2 = -20$ keV, $\kappa_4 = 0$, $q = 0.5$, $\eta_d = 1.5$, $\eta_g = 4.5$, and $N = 10$. Fig. 1 shows the effect of the truncation on low-lying band structure: a) band excitation energies, b) E2 transitions, and c) E4 transitions. The $n_{g\text{max}} = 1$ calculations are

off by about 10-20% (mostly overestimated but underestimated in a few cases), and hence they are not very reliable. As expected, the hexadecapole bands take longer to converge compared to the β and γ bands, the worst case being the β' band. Nevertheless, convergence to accuracy of a few percent is obtained in almost all cases for $n_{g\max} = 3$. In fig. 2, we present a similar study for the high-spin states in the ground band: a) excitation energies, and b) E2 transitions. At spins $L \sim 2N$, the $n_{g\max} = 1$ calculations are off by about 20-30% which will get even worse with increasing spin. The $n_{g\max} = 3$ results, on the other hand, provide a reasonably accurate picture up to spins $L \sim 2N$. Beyond that, g bosons start dominating the wave functions, and any truncation is likely to lead to substantial errors.

The above results suggest that diagonalization of the sdg-IBM Hamiltonians in a model space truncated to $n_{g\max} = N/3$ bosons will give a reliable description of states with spins $L < 2N$. This extends the applicability of the SDGBOSON code to $N = 14$ which covers roughly half of the deformed nuclei. It should nevertheless be emphasized that these computations are expensive, time consuming, and certainly not the best way to deal with the sdg-IBM problems. In the next section, we introduce the $1/N$ expansion which circumvents the shortcomings of numerical diagonalization.

4 $1/N$ expansion formalism for high-spin states

The $1/N$ expansion formalism [11] was developed as a response to difficulties in performing calculations in the sdg-IBM due to the inadequacy of group theoretical techniques and the large basis space problem in numerical diagonalization. It is based on angular momentum projected mean field theory and leads to analytic expressions for various physical quantities of interest. Initially, the $1/N$ calculations were carried out to order $1/N^2$ which is quite sufficient for low-lying states. An accurate description of high-spin states, on the other hand, requires inclusion of terms up to order $1/N^6$ which is not suitable for hand calculation. This difficulty has finally been overcome through the use of computer algebra. A brief description of the method for the ground band was given previously [13]. Here we provide details of the extended calculations including the results for the excited bands and

the electromagnetic transitions.

4.1 Ground band

We consider a general formulation of the IBM as this allows an elegant derivation of the $1/N$ formulas by fully exploiting the angular momentum algebra. Thus we introduce the boson creation and annihilation operators $b_{l\mu}^\dagger, b_{l\mu}$ with $l = 0, 2, 4, \dots$ where $b_0 = s$, $b_2 = d$, $b_4 = g$, etc. In order to keep the variational problem to a manageable size, it is necessary to assume that the boson system is axially symmetric, and hence K is a good quantum number. From comparison with the exact diagonalization results, this assumption will be seen to hold to a very good degree. The ground band then can be written as a condensate of intrinsic bosons as

$$|\phi_g\rangle = (N!)^{-1/2} (b^\dagger)^N |0\rangle, \quad b^\dagger = \sum_l x_l b_{l0}^\dagger, \quad (8)$$

where x_l are the normalized boson mean fields, i.e. $\mathbf{x} \cdot \mathbf{x} = 1$, $\mathbf{x} = (x_0, x_2, x_4, \dots)$. In the classical limit of the IBM, the mean fields are associated with the deformation parameters of the system [18]. For a given Hamiltonian H , they are determined from $\langle H \rangle_L$ by variation after projection (VAP).

The Hamiltonians in eqs. (1,4) can be written in the generalized form as

$$H = \sum_l \varepsilon_l \hat{n}_l - \sum_{k=0}^{2l_{\max}} \kappa_k T^{(k)} \cdot T^{(k)}, \quad \hat{n}_l = \sum_\mu b_{l\mu}^\dagger b_{l\mu}, \quad T^{(k)} = \sum_{jl} t_{kjl} [b_j^\dagger \tilde{b}_l]^{(k)}, \quad (9)$$

where the parameters have the obvious correspondence, $\varepsilon_2 = \varepsilon_d$, $\varepsilon_4 = \varepsilon_g$, $t_{2jl} = q_{jl}$, $t_{4jl} = h_{jl}$. This general form has the advantage that, to evaluate the expectation value of H , one needs to perform the calculation for a generic number operator \hat{n}_l and a multipole interaction $T^{(k)} \cdot T^{(k)}$. The expectation value of a scalar operator \hat{O} in the ground band (8), with angular momentum projection, is given by

$$\langle \hat{O} \rangle_L = \frac{2L+1}{2N! \mathcal{N}(\phi_g, L)} \int d\beta \sin \beta d_{00}^L(\beta) \langle 0 | b^N \hat{O} e^{-i\beta L_y} (b^\dagger)^N | 0 \rangle. \quad (10)$$

Here, the normalization, $\mathcal{N}(\phi_g, L)$, follows from eq. (10) upon substituting the identity operator for \hat{O} . Algebraic manipulations in eq. (10) are most easily carried out using boson calculus and angular momentum algebra techniques (see ref. [11] for a

pedagogical treatment). For the number operator, one obtains

$$\langle \hat{n}_l \rangle_L = \frac{N x_l^2}{F(N, L)} \sum_I \langle L 0 l 0 0 | I 0 \rangle^2 F(N-1, I), \quad (11)$$

where $F(N, L)$ denotes the reduced normalization integral

$$F(N, L) = \mathcal{N}(\phi_g, L)/(2L+1). \quad (12)$$

Eq. (11) is exact and highlights the essential role played by the normalization integral. In the original papers [11], a Gaussian approximation was used in the evaluation of $F(N, L)$ which limited the accuracy of m.e. to order $1/N^2$. This difficulty has been overcome recently using the computer algebra software Mathematica [19]. By exploiting the symmetries of the boson system, the normalization integral is cast into a system of linear equations which is solved with the help of Mathematica [20]. The result is a double expansion in $1/N$ and $\bar{L} = L(L+1)$ given by

$$F(N, L) = \frac{2}{aN} \sum_{n=0} \frac{(-1)^n}{n!(aN)^n} \sum_{m=0}^n \alpha_{nm} \bar{L}^m. \quad (13)$$

The coefficients α_{nm} in eq. (13) are given in terms of polynomials of the moments of x_l^2

$$a_n = \sum_l \bar{l}^{n+1} x_l^2, \quad (14)$$

and a is defined as $a \equiv a_0$. A list of α_{nm} up to sixth order is given in ref. [20]. The knowledge of $F(N, L)$, in principle, allows evaluation of m.e. to arbitrary orders in $1/N$. As will be seen in the applications, a correct description of MOI at high-spins requires inclusion of terms of order \bar{L}^3/N^6 . Evaluation of eq. (11) to such high orders is too difficult to perform by hand but becomes manageable using computer algebra.

Before presenting the final results, it will be useful to comment on the general form of the m.e. of a k -body operator \hat{O} , and illustrate the concept of layers in the $1/N$ expansion

$$\begin{aligned} \langle \hat{O} \rangle_L &= N^k \sum_{n,m} \frac{O_{nm}}{(aN)^m} \left(\frac{\bar{L}}{a^2 N^2} \right)^n \\ &= N^k \left\{ O_{00} + \frac{O_{01}}{aN} + \frac{O_{02}}{(aN)^2} + \frac{O_{03}}{(aN)^3} + \dots \right. \\ &\quad \left. + \frac{\bar{L}}{a^2 N^2} \left(O_{10} + \frac{O_{11}}{aN} + \frac{O_{12}}{(aN)^2} + \dots \right) \right\} \end{aligned}$$

$$\begin{aligned}
& + \left(\frac{\bar{L}}{a^2 N^2} \right)^2 \left(O_{20} + \frac{O_{21}}{aN} + \dots \right) \\
& + \left(\frac{\bar{L}}{a^2 N^2} \right)^3 \left(O_{30} + \dots \right) + \dots \}. \quad (15)
\end{aligned}$$

The expansion coefficients O_{nm} in eq. (15) involve various quadratic forms of the mean fields x_l corresponding to the single-boson m.e. of \hat{O} and its moments. The explicit form is given to facilitate the illustration of layers. Notice that the i coefficients O_{nm} in the i 'th column have $n + m = i - 1$ constant, and are referred as the layer “ $i - 1$ ”. The leading term in eq. (15) thus forms the zeroth layer. (This name is appropriate since calculations in the intrinsic frame give the same result independent of projection.) In simple terms, the layer in an expansion is given by the maximum power of \bar{L} . There is a close connection between the layers in the m.e. (15) and the normalization coefficients α_{nm} in eq. (13), namely, in order to calculate the m.e. up to the i 'th layer, one needs to know the coefficients $\{\alpha_{nn}, \alpha_{nn-1}, \dots, \alpha_{nn-i+1}, n = 1, 2i\}$. This is very useful in higher order calculations as it restricts the number of terms in the expansion, cutting down the amount of algebra. To make this point clear, we note that eq. (15) shows all the terms in the third layer whereas a complete calculation to order $1/N^6$ would require 6 more terms belonging to the fourth, fifth and sixth layers. As can be seen from eq. (16) below, the complexity of the coefficients O_{nm} increases “exponentially” with layers, and each of the extra terms would lead to expressions pages long. From a practical point of view, such accuracy is never required. The only $1/N^6$ term of any consequence is \bar{L}^3/N^6 which is included in the third layer. The rest are completely negligible. Hence use of layers is a more sensible approach than a complete calculation to a given order in $1/N$.

With these considerations, we present the result of the Mathematica evaluation of the one-body m.e. (11) to the third layer

$$\begin{aligned}
\langle \hat{n}_l \rangle_L = N x_l^2 & \left\{ 1 + \frac{1}{aN} (a - \bar{l}) + \frac{1}{(aN)^2} (-a + a_1/2 + (1 - a_1/a)\bar{l} + \bar{l}^2/2) \right. \\
& + \frac{1}{(aN)^3} \left(a + 2a^2 - 7a_1/3 - aa_1 + 5a_1^2/4a - a_2/3 \right. \\
& \quad \left. + (-1 - 2a + 2a_1 + 7a_1/2a - 5a_1^2/2a^2 + a_2/2a)\bar{l} \right. \\
& \quad \left. + (-7/6 - a + 5a_1/4a)\bar{l}^2 - \bar{l}^3/6 \right)
\end{aligned}$$

$$\begin{aligned}
& + \frac{\bar{L}}{(aN)^2} \left[(-a + \bar{l}) + \frac{1}{aN} (2a + 2a^2 - 2a_1 + (-2 - 2a + 3a_1/a)\bar{l} - \bar{l}^2) \right. \\
& \quad + \frac{1}{(aN)^2} \left(-3a - 12a^2 - 4a^3 + 21a_1/2 + 11aa_1 - 15a_1^2/2a + 3a_2/2 \right. \\
& \quad \quad + (3 + 12a + 4a^2 - 33a_1/2 - 14a_1/a + 25a_1^2/2a^2 - 2a_2/a)\bar{l} \\
& \quad \quad \left. \left. + (7/2 + 11a/2 - 5a_1/a)\bar{l}^2 + \bar{l}^3/2 \right) \right] \\
& + \frac{\bar{L}^2}{2(aN)^4} \left[(-a - 2a^2 + 3a_1/2 + (1 + 2a - 2a_1/a)\bar{l} + \bar{l}^2/2) \right. \\
& \quad + \frac{1}{aN} \left(4a + 21a^2 + 14a^3 - 16a_1 - 51aa_1/2 + 13a_1^2/a - 2a_2 \right. \\
& \quad \quad + (-4 - 21a - 14a^2 + 34a_1 + 20a_1/a - 39a_1^2/2a^2 + 5a_2/2a)\bar{l} \\
& \quad \quad \left. \left. + (-4 - 17a/2 + 13a_1/2a)\bar{l}^2 - \bar{l}^3/2 \right) \right] \\
& + \frac{\bar{L}^3}{3(aN)^6} \left[-a - 6a^2 - 6a^3 + 25a_1/6 + 9aa_1 - 15a_1^2/4a + 5a_2/12 \right. \\
& \quad + (1 + 6a + 6a^2 - 45a_1/4 - 5a_1/a + 21a_1^2/4a^2 - a_2/2a)\bar{l} \\
& \quad \left. \left. + (5/6 + 9a/4 - 3a_1/2a)\bar{l}^2 + \bar{l}^3/12 \right] \right\}, \tag{16}
\end{aligned}$$

where a_n is defined in eq. (14). Eq. (16) can be checked against two results: i) it satisfies the number conservation, i.e. $\sum_l \langle \hat{n}_l \rangle_L = N$, and ii) it reproduces the analytic formulas available in the SU(3) limit [1].

A similar calculation for the multipole interaction yields the intermediate result

$$\begin{aligned}
\langle T^{(k)} \cdot T^{(k)} \rangle_L = \frac{N(2k+1)}{F(N, L)} & \left\{ \sum_{jl} \frac{(t_{kjl}x_l)^2}{2l+1} \sum_I \langle L0l0|I0 \rangle^2 F(N-1, I) \right. \\
& + (N-1) \sum_{jlj'l'J} t_{kjl} t_{kj'l'} x_j x_l x_{j'} x_{l'} \langle j0j'0|J0 \rangle \langle l0l'0|J0 \rangle \\
& \left. \times \begin{Bmatrix} j & j' & J \\ l' & l & k \end{Bmatrix} \sum_I \langle L0J0|I0 \rangle^2 F(N-2, I) \right\}. \tag{17}
\end{aligned}$$

Again this is exact and can be evaluated to any order using Mathematica. The third layer result is given by

$$\begin{aligned}
\langle T^{(k)} \cdot T^{(k)} \rangle_L = N^2 & \left\{ U_k + \frac{1}{aN} (aU_k - U_{k1} + aC_k) \right. \\
& + \frac{1}{(aN)^2} \left((-2a + a_1)U_k + (1 - a - a_1/a)U_{k1} + U_{k2}/2 + a^2C_k - aC_{k1} \right) \\
& + \frac{1}{(aN)^3} \left((2a + 2a^2 - 14a_1/3 - aa_1 + 5a_1^2/2a - 2a_2/3)U_k \right. \\
& \quad \left. + (-1 + a - a_1/2 + 7a_1/2a - 5a_1^2/2a^2 + a_2/2a)U_{k1} \right.
\end{aligned}$$

$$\begin{aligned}
& +(-7/6 + 5a_1/4a)U_{k2} - U_{k3}/6 \\
& +(-a^2 + aa_1/2)C_k + (a - a_1)C_{k1} + aC_{k2}/2) \\
& + \frac{\bar{L}}{(aN)^2} \left[-2aU_k + U_{k1} \right. \\
& \quad + \frac{1}{aN} \left((4a + 2a^2 - 4a_1)U_k + (-2 + a + 3a_1/a)U_{k1} - U_{k2} - a^2C_k + aC_{k1} \right) \\
& \quad + \frac{1}{(aN)^2} \left((-6a - 16a^2 - 4a^3 + 21a_1 + 15aa_1 - 15a_1^2/a + 3a_2)U_k \right. \\
& \quad \quad + (3 + 2a - 2a^2 - 4a_1 - 14a_1/a + 25a_1^2/2a^2 - 2a_2/a)U_{k1} \\
& \quad \quad + (7/2 + 2a - 5a_1/a)U_{k2} + U_{k3}/2 \\
& \quad \quad \left. + (2a^2 + 2a^3 - 2aa_1)C_k + (-2a - 2a^2 + 3a_1)C_{k1} - aC_{k2} \right) \Big] \\
& + \frac{\bar{L}^2}{2(aN)^4} \left[(-2a - 2a^2 + 3a_1)U_k + (1 - 2a_1/a)U_{k1} + U_{k2}/2 \right. \\
& \quad + \frac{1}{aN} \left((8a + 30a^2 + 14a^3 - 32a_1 - 37aa_1 + 26a_1^2/a - 4a_2)U_k \right. \\
& \quad \quad + (-4 - 8a + 2a^2 + 29a_1/2 + 20a_1/a - 39a_1^2/2a^2 + 5a_2/2a)U_{k1} \\
& \quad \quad + (-4 - 9a/2 + 13a_1/2a)U_{k2} - U_{k3}/2 \\
& \quad \quad \left. + (-a^2 - 2a^3 + 3aa_1/2)C_k + (a + 2a^2 - 2a_1)C_{k1} + aC_{k2}/2 \right) \Big] \\
& + \frac{\bar{L}^3}{3(aN)^6} \left[(-8a - 36a^2 - 24a^3 + 100a_1/3 + 54aa_1 - 30a_1^2/a + 10a_2/3)U_k \right. \\
& \quad + (4 + 12a - 24a_1 - 20a_1/a + 21a_1^2/a^2 - 2a_2/a)U_{k1} \\
& \quad \left. + (10/3 + 6a - 6a_1/a)U_{k2} + U_{k3}/3 \right] \Big\}. \tag{18}
\end{aligned}$$

Here the quadratic forms C_{kn} arise from normal ordering and simulate an effective one-body term

$$C_{kn} = (2k+1) \sum_{jl} \bar{l}^n (t_{kjl}x_l)^2 / (2l+1), \tag{19}$$

while U_{kn} represent the genuine two-boson interaction

$$U_{kn} = \sum_{jlj'l'I} \bar{I}^n \langle j0j'0|I0 \rangle \langle l0l'0|I0 \rangle \left\{ \begin{matrix} j & j' & I \\ l' & l & k \end{matrix} \right\} t_{kjl} t_{kjl'} x_j x_l x_{j'} x_{l'}. \tag{20}$$

For a given multipole, these sums can be evaluated in closed form using Mathematica. For the quadrupole and hexadecapole interactions, the first four terms needed in eq. (18) are given by

$$U_2 = A^2,$$

$$\begin{aligned}
U_{21} &= (2A_1 - 3A)A, \\
U_{22} &= (2A_2 - 24A_1 + 18A)A + (A_{11} - A_2 + 7A_1)A_1 + (A_{11} - A_2)^2/12, \\
U_{23} &= (2A_3 - 36A_2 - 18A_{11} + 240A_1 - 144A)A \\
&\quad + (3A_{21} - 3A_3 + 56A_2 + 16A_{11} - 194A_1)A_1/2 \\
&\quad + (11A_{11}^2 + 14A_{11}A_2 - 25A_2^2)/12 + (A_3 - A_{21})(A_2 - A_{11})/4, \\
U_4 &= B^2, \\
U_{41} &= (2B_1 - 10B)B/2, \\
U_{42} &= 4B_1^2 + (2B_{11} - 40B_1 + 20B)B + (B_2 - B_{11} - 20B_1 + 180B)^2/180 \\
U_{43} &= (2B_3 - 120B_2 - 60B_{11} + 2480B_1 - 4400B)B \\
&\quad + (3B_{21} - 3B_3 + 224B_2 + 58B_{11} - 2756B_1/3)B_1/9 \\
&\quad + (8B_{11}^2 + 14B_{11}B_2 - 11B_2^2)/45 + (B_3 - B_{21})(B_2 - B_{11})/60 \tag{21}
\end{aligned}$$

Here the quadratic forms A_{mn} and B_{mn} in (21) are defined as

$$A_{mn} = \sum_{jl} \bar{j}^m \bar{l}^n \langle j0l0|20 \rangle t_{2jl} x_j x_l, \quad B_{mn} = \sum_{jl} \bar{j}^m \bar{l}^n \langle j0l0|40 \rangle t_{4jl} x_j x_l, \tag{22}$$

and correspond to various moments of the single-boson m.e. of the quadrupole and hexadecapole operators. (note that the zero subscripts are suppressed for convenience). The quadrupole m.e. given by (18-21) reproduces the well known Casimir eigenvalues in the SU(3) limit, hence also passes the SU(3) test.

The analytic expressions presented above are already rather long. If for any reason, the next layer results should be required, the expressions would grow to pages long, and the analytical $1/N$ calculations might not be very practical. In such cases, numerical evaluation of the m.e. (11,17), as described in Appendix A, may be preferable. Although this would increase the computation time appreciably, it has the advantage that the calculations are done exactly to all orders in $1/N$.

4.2 Variation after projection

The energy expression derived in the last subsection is rather lengthy, and in discussing the variational problem, it will be more convenient to express it in a compact form. Thus, using the parametrization in eq. (7), we rewrite the ground band energy

as

$$E_{gL} = N^2 \kappa_2 \sum_{n,m} \frac{E_{nm}}{N^m} \left(\frac{\bar{L}}{N^2} \right)^n, \quad (23)$$

where the coefficients E_{nm} can be read off from eqs. (16-22). For example the leading order is given by

$$E_{00} = \sum_l \eta_l \frac{x_l^2}{\mathbf{x} \cdot \mathbf{x}} - \left(\frac{A}{\mathbf{x} \cdot \mathbf{x}} \right)^2 - \zeta_4 \left(\frac{B}{\mathbf{x} \cdot \mathbf{x}} \right)^2, \quad (24)$$

with A and B defined in eq. (22), and we have restored the normalization factors $\mathbf{x} \cdot \mathbf{x}$ as a precursor to variation. The minimum of the ground energy is obtained from

$$\partial E_{gL} / \partial x_l = 0, \quad l = 0, 2, 4, \dots, \quad (25)$$

which can be solved algebraically using the ansatz

$$\mathbf{x} = \sum_{n,m} \frac{\mathbf{x}_{nm}}{N^m} \left(\frac{\bar{L}}{N^2} \right)^n. \quad (26)$$

The use of layers again simplifies solution of the variational equations. For the leading order (zeroth layer), one has the usual Hartree-Bose equations

$$\left. \frac{\partial E_{00}}{\partial x_l} \right|_{\mathbf{x}_{00}} = 0, \quad (27)$$

which are a system of coupled non-linear equations, and they are solved numerically by iteration [21]. Having determined \mathbf{x}_{00} , the first layer mean fields \mathbf{x}_{01} and \mathbf{x}_{10} are then obtained by solving the respective sets of equations

$$\begin{aligned} \left. \frac{\partial E_{00}}{\partial x_l} \right|_{\mathbf{x}_{00} + \mathbf{x}_{01}/N} &= -\frac{1}{N} \left. \frac{\partial E_{01}}{\partial x_l} \right|_{\mathbf{x}_{00}}, \\ \left. \frac{\partial E_{00}}{\partial x_l} \right|_{\mathbf{x}_{00} + \mathbf{x}_{10}\bar{L}/N^2} &= -\frac{\bar{L}}{N^2} \left. \frac{\partial E_{10}}{\partial x_l} \right|_{\mathbf{x}_{00}}. \end{aligned} \quad (28)$$

Upon substituting the mean fields in derivatives in (28), the leading order vanishes by virtue of the Hartree-Bose eqs. (27), and the next order leads to sets of linear equations for \mathbf{x}_{01} and \mathbf{x}_{10} , that can be easily solved using Mathematica. The Hartree-Bose condition also ensures that when the first layer mean fields are substituted in the energy expression, the correction to the first layer exactly vanishes. So they only contribute to the second and higher layers [21]. This holds in general for all layers.

Thus for the third layer expansion considered here, one needs at most the second layer mean fields \mathbf{x}_{02} , \mathbf{x}_{11} and \mathbf{x}_{20} which are obtained from

$$\begin{aligned}
\left. \frac{\partial E_{00}}{\partial x_l} \right|_{\mathbf{x}_{00}+\mathbf{x}_{01}/N+\mathbf{x}_{02}/N^2} &= -\frac{1}{N} \left. \frac{\partial E_{01}}{\partial x_l} \right|_{\mathbf{x}_{00}+\mathbf{x}_{01}/N} - \frac{1}{N} \left. \frac{\partial E_{02}}{\partial x_l} \right|_{\mathbf{x}_{00}}, \\
\left. \frac{\partial E_{00}}{\partial x_l} \right|_{\mathbf{x}_{00}+\mathbf{x}_{10}\bar{L}/N^2+\mathbf{x}_{20}\bar{L}^2/N^4} &= -\frac{\bar{L}}{N^2} \left. \frac{\partial E_{10}}{\partial x_l} \right|_{\mathbf{x}_{00}+\mathbf{x}_{10}\bar{L}/N^2} - \frac{\bar{L}^2}{N^4} \left. \frac{\partial E_{20}}{\partial x_l} \right|_{\mathbf{x}_{00}}, \\
\left. \frac{\partial E_{00}}{\partial x_l} \right|_{\mathbf{x}_{00}+\mathbf{x}_{01}/N+\mathbf{x}_{10}\bar{L}/N^2+\mathbf{x}_{11}\bar{L}/N^3} &= -\frac{1}{N} \left. \frac{\partial E_{01}}{\partial x_l} \right|_{\mathbf{x}_{00}+\mathbf{x}_{10}\bar{L}/N^2} - \frac{\bar{L}}{N^2} \left. \frac{\partial E_{10}}{\partial x_l} \right|_{\mathbf{x}_{00}+\mathbf{x}_{01}/N} \\
&\quad - \frac{\bar{L}}{N^3} \left. \frac{\partial E_{11}}{\partial x_l} \right|_{\mathbf{x}_{00}}, \tag{29}
\end{aligned}$$

Again these sets of linear equations can be solved using Mathematica. We refrain from presenting these rather bulky results for the first and second layer mean fields here because, in the absence of analytical solutions for the zeroth layer, they are not very illuminating. Upon substituting eq. (26) in (23), one obtains the variational corrections introduced by the higher order mean fields in the ground band energies. These lengthy analytic expressions contribute only to the second and higher layers and will not be shown here. All these results, together with other $1/N$ expansion formulas, are nevertheless available in the form of a Fortran code [22]. Finally, if one is interested only in practical applications of the results to high-spin states, one can determine the minimum directly from the energy expression (23) using the numerical simplex method, and thereby avoid the complexities introduced by the higher order terms in the solution of the variational problem.

4.3 Single-phonon bands

Most of the high-spin data, as well as their theoretical analysis, are concentrated on the yrast bands (ground or two-quasiparticle). While relying solely on the yrast data may be tolerated for microscopic models, it could easily lead to misleading results in phenomenological models. For this reason, inclusion of single-phonon bands in the analysis of high-spin data is highly desirable in phenomenological approaches. As will be seen in the applications, there are substantial high-spin data for the γ bands, which can be singled out among the single-phonon bands in this respect. Therefore, we consider here the γ band as an example of a single-phonon band calculation.

Energy expressions for the other bands can be derived in a similar fashion.

The single-phonon bands are obtained from the ground band by acting with the other intrinsic boson operators $b_m^\dagger = \sum_l x_{lm} b_{lm}^\dagger$, and then orthogonalizing the resulting bands. For example, the γ band intrinsic state is given by

$$|\phi_\gamma\rangle = b_2^\dagger |\phi_g, N-1\rangle + \frac{1}{\sqrt{N-1}} \xi_\gamma (b_1^\dagger)^2 |\phi_g, N-2\rangle, \quad (30)$$

In this trial state, the mean fields for the ground band are already established in the last subsection, and those for b_1 are determined from the spurious $K=1$ band as [11],

$$x_{l1} = [\bar{l}/a]^{1/2} x_l. \quad (31)$$

Thus, only the γ band mean fields, x_{l2} , are to be determined by VAP. The coefficient ξ_γ in eq. (30) follows from the orthogonality condition $\langle L_\gamma | L_g \rangle = 0$ as

$$\begin{aligned} \sum_{ll'} \langle L2l-2 | I0 \rangle \langle L0l0 | I0 \rangle & \left[x_l x_{l2} F(N-1, I) \right. \\ & \left. + \xi_\gamma \sum_{jj'} x_j x_{j1} x_{j'} x_{j'1} \langle j0j'0 | l0 \rangle \langle j1j'1 | l2 \rangle F(N-2, I) \right] = 0 \end{aligned} \quad (32)$$

where F denotes the ground band normalization (13) for $N-1$ and $N-2$ bosons.

The expectation value of a scalar operator \hat{O} in the γ band (30), with angular momentum projection, is given by

$$\begin{aligned} \langle \hat{O} \rangle_{\gamma, L} = \frac{2L+1}{2(N-1)\mathcal{N}(\phi_\gamma, L)} \int d\beta \sin \beta d_{22}^L(\beta) & \left\{ \langle 0 | b^{N-1} b_2 \hat{O} e^{-i\beta L_y} (b^\dagger)^{N-1} b_2^\dagger | 0 \rangle \right. \\ & + 2\xi_\gamma \langle 0 | b^{N-2} b_1^2 \hat{O} e^{-i\beta L_y} (b^\dagger)^{N-1} b_2^\dagger | 0 \rangle \\ & \left. + \xi_\gamma^2 \langle 0 | b^{N-2} b_1^2 \hat{O} e^{-i\beta L_y} (b^\dagger)^{N-2} (b_1^\dagger)^2 | 0 \rangle \right\}, \end{aligned} \quad (33)$$

where $\mathcal{N}(\phi_\gamma, L)$ is the normalization for the γ band, obtained from eq. (33) using the identity operator for \hat{O} . The contribution from the orthogonality terms to the band energies are of the order $1/N^2$, and therefore they were ignored in the original papers [11]. In description of high-spin states, however, these terms make essential contributions and they have to be included in the calculations. Each contraction of the intrinsic boson operators in (33) leads to projected single-boson overlaps of the form $x_{lm} x_{lm'} d_{mm'}^l$. The resulting Wigner d -functions are coupled to a final d -function to perform the β integral. This process leads to rather long expressions

for the orthogonality terms. In order to reduce their size, we introduce a compact notation for the recoupling coefficients as follows

$$\begin{aligned}
R_2(jmm', lnn'; I) &= x_{jm}x_{jm'}x_{ln}x_{ln'}\langle jmln|Im+n\rangle\langle jmln'|Im'+n'\rangle, \\
R_3(jmm', lnn', k\mu\mu'; I, J) &= R_2(jmm', lnn'; I)x_{k\mu}x_{k\mu'}\langle Im+nk\mu|Jm+n+\mu\rangle \\
&\quad \langle Im'+n'k\mu'|Jm'+n'+\mu'\rangle, \\
R_4(jmm', lnn', k\mu\mu', k'\nu\nu'; I, I', J) &= R_2(jmm', lnn'; I)R_2(k\mu\mu', k'\nu\nu'; I') \\
&\quad \langle Im+nI'\mu+\nu|Jm+n+\mu+\nu\rangle \\
&\quad \langle Im'+n'I'\mu'+\nu'|Jm'+n'+\mu'+\nu'\rangle. \quad (34)
\end{aligned}$$

Higher recoupling coefficients (R_5, R_6) are defined similarly. Using this notation, the reduced normalization for the γ band, $F_\gamma(N, L) = \mathcal{N}(\phi_\gamma, L)/(2L+1)$, can be written as

$$\begin{aligned}
F_\gamma(N, L) &= \sum_{Ij} \langle L2j-2|I0\rangle^2 \\
&\quad \times \left\{ x_{j2}^2 F(N-1, I) + (N-1) \sum_{ll'} R_2(l'20, l02; j) F(N-2, I) \right. \\
&\quad + 2\xi_\gamma \left[2 \sum_{ll'} R_2(l'21, l01; j) F(N-2, I) \right. \\
&\quad \left. + (N-2) \sum_{kk'll'} R_3(k10, k'10, l02; l', j) F(N-3, I) \right] \\
&\quad + \frac{2\xi_\gamma^2}{N-1} \left[\sum_{ll'} R_2(l'11, l11; j) F(N-2, I) \right. \\
&\quad + 2(N-2) \sum_{kk'll'} R_3(k10, k'01, l11; l', j) F(N-3, I) \\
&\quad + \frac{1}{2}(N-2)(N-3) \sum_{kk'k''ll'l''} R_4(k10, k'10, l01, l'01; k'', l'', j) \\
&\quad \left. \left. \times F(N-4, I) \right] \right\}. \quad (35)
\end{aligned}$$

Eq. (35) expresses the γ -band normalization in terms of the ground-band normalization (13), and it can be evaluated to any order in $1/N$ using Mathematica.

The expectation value of the number operator in the γ band can be calculated similarly, giving

$$\langle \hat{n}_l \rangle_{\gamma, L} = \frac{1}{F_\gamma(N, L)} \sum_{Ij} \langle L2j-2|I0\rangle^2 \left\{ x_{l2}^2 \delta_{jl} F(N-1, I) \right.$$

$$\begin{aligned}
& +(N-1) \sum_{l'} \left(R_2(l'22, l00; j) + 2R_2(l'20, l02; j) \right) F(N-2, I) \\
& +(N-1)(N-2) \sum_{kk'l'} R_3(k20, k'02, l00; l', j) F(N-3, I) \\
& +2\xi_\gamma \left[2 \sum_{l'} \left(R_2(l'21, l01; j) + R_2(l'01, l21; j) \right) F(N-2, I) \right. \\
& \quad + (N-2) \sum_{kk'l'} \left(2R_3(k21, k'01, l00; l', j) + 2R_3(k20, k'01, l01; l', j) \right. \\
& \quad \quad \left. \left. + R_3(k10, k'10, l02; l', j) \right) F(N-3, I) \right. \\
& \quad \left. + (N-2)(N-3) \sum_{kk'k''l'l''} R_4(k10, k'10, l'02, l00; k'', l'', j) F(N-4, I) \right] \\
& + \frac{2\xi_\gamma^2}{N-1} \left[2 \sum_{l'} R_2(l'11, l11; j) F(N-2, I) \right. \\
& \quad + (N-2) \sum_{kk'l'} \left(R_3(k11, k'11, l00; l', j) + 4R_3(k11, k'10, l01; l', j) \right. \\
& \quad \quad \left. + 2R_3(k10, k'01, l11; l', j) \right) F(N-3, I) \\
& \quad + 2(N-2)(N-3) \sum_{kk'k''l'l''} \left(R_4(k11, k'10, l'01, l00; k'', l'', j) \right. \\
& \quad \quad \left. + R_4(k10, k'10, l'01, l01; k'', l'', j) \right) F(N-4, I) \Big] \Big\}. \quad (36)
\end{aligned}$$

It can be easily checked that the condition, $\sum_l \langle \hat{n}_l \rangle_{\gamma, L} = N$ is satisfied by (36). The expectation value of a general two-body interaction in the γ band is given by

$$\begin{aligned}
\langle T^k \cdot T^k \rangle_{\gamma, L} &= \sum_{jl} \frac{2k+1}{2l+1} t_{kjl}^2 \langle \hat{n}_l \rangle_{\gamma, L} \\
&+ \frac{(N-2)(2k+1)}{F_\gamma(N, L)} \sum_{jlj'l'J} t_{kjl} t_{kj'l'} \left\{ \begin{matrix} j' & j & J \\ l & l' & k \end{matrix} \right\} \sum_{iII'} \langle L2I' - 2 | I0 \rangle^2 \\
&\times \left\{ (N-1) \left[\frac{2}{N-2} (P_{2002} + P_{0022}) F(N-2, I) \delta_{i0} \right. \right. \\
&\quad + \left(x_{i2}^2 P_{0000} + 4x_{i2} x_i P_{2000} \right) F(N-3, I) \\
&\quad \left. + (N-3) \sum_{kk'} R_2(k20, k'02; i) P_{0000} F(N-4, I) \right] \\
&+ 4\xi_\gamma \left[\frac{2}{N-2} P_{2101} \delta_{i0} F(N-2, I) \right. \\
&\quad + \left(2x_{i2} x_{i1} P_{0001} + x_{i2} x_i P_{0101} + 2x_{i1} x_i (P_{2001} + P_{0021}) \right) F(N-3, I) \\
&\quad \left. + (N-3) \sum_{kk'} \left(R_2(k21, k'01; i) P_{0000} + 2R_2(k20, k'01; i) P_{0001} \right. \right.
\end{aligned}$$

$$\begin{aligned}
& +R_2(k10, k'10; i)P_{0002})F(N-4, I) \\
& +\frac{1}{2}(N-3)(N-4) \sum_{kk'k''i'} R_3(k10, k'10, i'02; k'', i)P_{0000}F(N-5, I) \Big] \\
& +\frac{4\xi_\gamma^2}{N-1} \Big[\frac{1}{N-2}P_{1111}F(N-2, I) \\
& +2\left(x_{i1}^2(P_{1001} + P_{0011}) + 2x_{i1}x_iP_{1011}\right)F(N-3, I) \\
& +(N-3) \sum_{kk'} \left(\frac{1}{2}R_2(k11, k'11; i)P_{0000} + 4R_2(k11, k'10; i)P_{0001} \right. \\
& \quad \left. +2R_2(k10, k'01; i)(P_{1001} + P_{0011}) + R_2(k10, k'10; i)P_{0101}\right)F(N-4, I) \\
& +(N-3)(N-4) \sum_{kk'k''i'} \left(R_3(k11, k'10, i'01; k'', i)P_{0000} \right. \\
& \quad \left. +2R_3(k10, k'10, i'01; k'', i)P_{0001}\right)F(N-5, I) \\
& +\frac{1}{4}(N-3)(N-4)(N-5) \sum_{kk'k''i'i''l''} R_4(k10, k'10, i'01, i''01; k'', l'', i) \\
& \quad \left. \times P_{0000}F(N-6, I) \Big] \Big\}. \tag{37}
\end{aligned}$$

Here we have introduced the compact notation for the two-boson m.e.

$$\begin{aligned}
P_{m'nmm'n'} &= x_{jm}x_{j'm'}x_{ln}x_{l'n'}\langle j'm'jm|Jm+m'\rangle\langle lnl'n'|Jn+n'\rangle \\
&\quad \langle Jm+m'i2-m-m'|I'2\rangle\langle Jn+n'i2-n-n'|I'2\rangle. \tag{38}
\end{aligned}$$

The dummy summation indices j, j', l, l', J, i, I' in P are suppressed for convenience. The first term in eq. (37) is the effective one-body term that arises from normal ordering of the boson operators in the multipole interaction, and it is expressed using eq. (36).

Eqs. (36,37) are the counterparts of eqs. (11,17) for the ground band and can be fed directly into Mathematica for evaluation. As one can surmise from a cursory comparison of the parent equations, the resulting third layer expressions are pages long. They are not as accurate as the ground band results, presumably requiring inclusion of even higher order terms. For these reasons, we have opted for a numerical evaluation of eqs. (36,37) in the applications. Such a calculation includes all orders in $1/N$, and hence provides more reliable results for the γ -band energies.

4.4 E2 transitions

Description of the yrast E2 transitions at high-spins is one of the main aims of this work. Therefore, we present a brief review of the currently available $1/N$ results for E2 m.e. and discuss their extensions to higher orders. A comprehensive study of the E2 transitions among the ground, γ and β bands was given previously [23]. The first layer m.e. obtained in [23] for the yrast E2 transitions appears to work rather well even at high-spins [13]. Inclusion of the d-boson energy leads to some deterioration at high-spins, which can be rectified by incorporating the higher order terms in the expansion.

The ground band m.e. of the quadrupole operator is given by

$$\begin{aligned} \langle L' \parallel Q \parallel L \rangle &= \hat{L} [4F(N, L')F(N, L)]^{-1/2} \sum_M \langle LM2 - M | L'0 \rangle \\ &\times \int d\beta \sin \beta d_{M0}^L \langle 0 | b^N Q_{-M} e^{-i\beta L_y} (b^\dagger)^N | 0 \rangle, \end{aligned} \quad (39)$$

where $\hat{L} = [2L + 1]^{1/2}$. As before, this can be reduced to the form

$$\begin{aligned} \langle L' \parallel Q \parallel L \rangle &= \frac{\sqrt{5}N\hat{L}\hat{L}'}{[F(N, L')F(N, L)]^{1/2}} \sum_{jlJ} q_{jl} x_j x_l \\ &\times \langle j0L'0 | J0 \rangle \langle L0l0 | J0 \rangle \left\{ \begin{matrix} j & L' & J \\ L & l & 2 \end{matrix} \right\} F(N - 1, J), \end{aligned} \quad (40)$$

which can be evaluated to any order using Mathematica. Because of the tensor nature of the E2 operator, however, the resulting third layer expressions are much more complicated than those for the Hamiltonian. In contrast, because there is no variation involved, numerical evaluation of eq. (40) is straightforward and is preferred over the lengthy algebraic forms in the following.

4.5 Comparison with the exact results

Before applying the $1/N$ expansion results, we compare them with those obtained from an exact diagonalization of the Hamiltonian [17]. Of necessity, the boson number is fixed at $N = 10$. The Hamiltonian parameters are as in sect. 3 ($\kappa_2 = -20$ keV, $\kappa_4 = 0$, $q = 0.5$, $\eta_d = 1.5$, $\eta_g = 4.5$), except where noted. Fig. 3 shows a comparison of the ground band energies normalized with $\bar{L} = L(L + 1)$ so that they all have the same energy scale. Fig. 3a studies the convergence of the $1/N$

results obtained with the VAP procedure. The second layer $1/N$ results (dotted line) rapidly diverge from the exact energies (circles) for spins $L > 2N$, and hence are not reliable in applications to high-spin states. The third layer results (dashed line), on the other hand, track the exact energies within a few percent up to the maximum spin $L = 4N$. Using the numerical technique described in Appendix A, one can evaluate the $1/N$ expansion to all orders (solid line) which exhibits an almost perfect agreement with the exact energies. This study demonstrates that the third layer $1/N$ expansion results are both necessary and sufficient for a reliable description of high-spin states. Fig. 3b shows the effect of the one-body energies on the accuracy of the third layer results. The top line is as in fig. 3a, the middle one compares the exact and $1/N$ results for $\eta_d = 0$, $\eta_g = 4.5$, and the bottom one for $\eta_d = 0$, $\eta_g = 0$. It is seen that the agreement for a pure quadrupole Hamiltonian is excellent at all spins, while the addition of g-boson energy leads to a few percent deviation at very high-spins. In a typical situation with d-boson energy, this few percent deviation starts occurring at medium high-spins.

In fig. 4a, we present a similar study for the gamma band energies. The average behaviour is well reproduced by the $1/N$ expansion results but staggering is underestimated. This happens because staggering is caused mainly by band mixing between the ground and γ bands which is not included in the present calculations (note that the odd-spin levels, which are not affected by band mixing, are very well reproduced). We have not attempted to include band mixing effects here because they are strongly suppressed for the larger N values used in deformed nuclei [23], and hence they can be ignored for the purposes of this work. Finally, in fig. 4b, we compare the $1/N$ results for the yrast $E2$ transition m.e. with the exact ones (circles). The dashed line shows the first layer result obtained from eq. (42) which is accurate to a few percent for $L < 2N$, but progressively gets worse with increasing spin. Solid line shows the numerical evaluation of eq. (40), which is complete to all orders in $1/N$. The agreement with the exact results becomes almost perfect in this case, including the highest spins which are dominated by g bosons.

5 Systematic studies

Application of the IBM to deformed nuclei has been criticized on a number of fronts ranging from the energy scale of γ excitation modes to the spin dependence of MOI and E2 transitions [3]. Most of these problems have been attributed to the truncation of the model space to s and d bosons but due to lack of adequate tools they have not been properly addressed in the framework of the sdg-IBM. The analytic formulas obtained in the last section have the advantage that one can easily perform systematic studies of key physical quantities and obtain useful insights on the effect of various parameters. In this section, we present such a study that will shed light on the above problems and suggest more appropriate Hamiltonians for description of deformed nuclei. As mentioned in sect. 2, the minimal sdg-IBM Hamiltonian with quadrupole interaction and g-boson energy has previously been applied to deformed nuclei and found to be inadequate on most counts [12, 14]. Inclusion of the d-boson energy and hexadecapole interaction appear to be necessary to improve this situation. Here we focus on the effect of these extra terms on the minimal Hamiltonian, especially with regard to high-spin states.

To simplify the discussion, we rewrite the ground and γ band energies as

$$E_{gL} = \lambda_{g1}\bar{L} + \lambda_{g2}\bar{L}^2 + \lambda_{g3}\bar{L}^3, \quad E_{\gamma L} = E_{\gamma} + \lambda_{\gamma1}\bar{L} + \lambda_{\gamma2}\bar{L}^2 + \lambda_{\gamma3}\bar{L}^3, \quad (41)$$

where the coefficients λ_n can be read from the respective energy expressions. Eq. (41) is the familiar rotational expansion of the level energies used in the geometrical model [24]. The difference between the two models is that in the IBM the coefficients λ_n follow from an underlying Hamiltonian (which is used in describing other properties) whereas in the geometrical model they are directly extracted from the data. The MOI problem raised in Ref. [3] refers to the fact that i) the λ_1 coefficient gets a substantial contribution from the dipole interaction, $L \cdot L$, which has no dynamical content, ii) the λ_2 coefficient is much smaller than the experimental values, and iii) the variation in λ_1 among different bands can not be described. All three problems are in fact interrelated. Although the second can be resolved by renormalizing the moment of inertia at high-spins (e.g. by modifying $L \cdot L \rightarrow L \cdot L/(1 + fL \cdot L)$ [25]), such modifications are purely kinematical in origin and do not address the

dynamical problem. Further quantities of interest in the study of high-spin states are the yrast E2 transitions. For systematics, it is sufficient to consider the first layer $1/N$ expansion result which has the generic form

$$\langle L-2 \parallel T(E2) \parallel L \rangle = e_2 N \hat{L} \langle L020 | L-20 \rangle [m_1 + m_2 L(L-1)] \quad (42)$$

where the coefficients m_n are given in ref. [23]. The first term in (42) gives the familiar rigid-rotor result. The second term is negative and is responsible for the falloffs predicted in E2 transitions.

In presenting systematics, we find it convenient to use ratios which eliminate the undesired effects of the scale parameters κ_2 and N . The energy scale can be fixed, for example, by fitting κ_2 to the excitation energy of the γ band, E_γ . We discuss five such ratios as a function of q for various values of (a) $\eta_d = \varepsilon_d/N\kappa_2$ and (b) $\zeta_4 = \kappa_4/\kappa_2$. The parameter q is varied from 0-1 which covers the whole range of the quadrupole operator from the γ -unstable to the SU(3) limit. η_d is varied from 0-2 in 10 equal steps, and ζ_4 from 0-0.5 in 5 equal steps, which cover the range of values used in the applications. (Negative values of ζ_4 on the whole are found to have an adverse effect and hence are not considered.) In the η_d systematics study, $\zeta_4 = 0$ is used as its precise value does not have much influence on the results. In the ζ_4 study, however, the choice of η_d does have an impact, and we adapt $\eta_d = 1.5$ which is the average value used in the applications. The g-boson energy has been found to have negligible effect [14], and therefore is not varied here but fixed at $\eta_g = 4.5$. Below, we comment on the behaviour of each ratio and contrast them with the experimental data. For reference, we note that q assumes values around ~ 0.5 in the rare-earth nuclei and ~ 0.7 in the actinides.

1) $E_\gamma/N\lambda_{g1}$ (fig. 5): This ratio relates the energy scales of the γ and ground bands, and its mismatch with experiment has been a source of criticism [3]. It is around 4-5 in the rare-earth region and increases to 8-9 in the actinides. From fig. 5a it is seen that the CQF with $\eta_d = 0$ overestimates it by about a factor of 2 in both regions. It decreases rapidly with η_d though, and through a judicious use of the d-boson energy, it should be possible to describe this ratio (and hence the MOI) without appealing to the $L \cdot L$ term. Fig. 5b shows a similar study on the effect of

the hexadecapole interaction which is seen to be going in the right direction but is too small to have any impact.

2) $N^2\lambda_{g2}/\lambda_{g1}$ (fig. 6): This ratio measures the deviation from the rigid rotor behaviour due to loss of pairing. It ranges from about -0.2 in the rare-earth region to -0.1 in the actinides. The CQF with $\eta_d = 0$ gives values an order of magnitude smaller (fig. 6a) and hence signally fails in accounting for the spin dependence of MOI as first pointed out in ref. [3]. However, this ratio is very sensitive to the η_d values and the experimental range can be easily attained by including the d-boson energy in the Hamiltonian. From fig. 6b, the hexadecapole interaction is seen to have a coherent effect in further reducing this ratio away from the rigid rotor behaviour.

3) $N^2\lambda_{\gamma2}/\lambda_{\gamma1}$ (fig. 7): An identical study for the γ band indicates broadly similar but somewhat larger effects of the d-boson energy on the behaviour of γ band MOI (fig. 7a). A softer MOI in the γ band is in line with data in most deformed nuclei though there are a few exceptions as will be seen in the applications. The hexadecapole interaction has an opposite effect (fig. 7b) which reduces the difference between the ground and γ band MOI caused by the d-boson energy.

4) $\lambda_{\gamma1}/\lambda_{g1}$ (fig. 8): This ratio compares the MOI of ground and γ bands which fluctuates within a band of $\pm 10\%$ across the deformed nuclei. The earlier IBM calculations gave results near one and could not accommodate such fluctuations. Fig. 8a-b shows that inclusion of the d-boson energy can increase this ratio by up to 20-30%, while the hexadecapole interaction can reduce it significantly (up to 20-30%) thereby covering the whole range of fluctuations.

5) N^2m_2/m_1 (fig. 9): As there is no boson cutoff effect, experimentally this ratio is consistent with zero. The CQF with $\eta_d = 0$ leads to values similar to the sd-IBM (fig. 9a), hence despite the addition of g bosons, it would suffer from the same boson cutoff problem. Introduction of the d-boson energy, however, reduces it substantially, becoming more in line with experiments. The effect of the hexadecapole interaction on this ratio (fig. 9b) is similar to fig. 5b; it is positive but comparatively too small to make a difference.

Another ratio, namely, $N^4\lambda_{g3}/\lambda_{g1}$ is also of interest especially at very high-spins ($L = 20 - 30$) where the cubic term in eq. (41) plays an important role [14]. It

exhibits a similar dependence on η_d and ζ_4 as $N^2\lambda_{g2}/\lambda_{g1}$ in fig. 6, so it is not discussed further here.

The d-boson energy has been mostly neglected in studies of deformed nuclei, presumably due to the success of the CQF with $\varepsilon_d = 0$ in explaining the energy and E2 transition systematics of low-lying states [6]. In fact, for small values ($\eta_d \sim 1$), its effect on low-lying states is negligible and it is not really needed in their description [26]. The CQF, however, basically leads to a rigid MOI and can not explain either its spin dependence or its variation among different bands. The obvious way towards a softer energy surface is to include the d-boson energy in the Hamiltonian which is seen to vastly improve the description of the spin-dependent terms in the level energies and E2 transitions. The hexadecapole interaction performs a similar function but has a much smaller effect. The exception is, of course, variations in MOI which could not be reproduced without the hexadecapole interaction.

6 Applications to deformed nuclei

In the light of the systematic trends discussed above, we carry out fits to the rare-earth nuclei $^{158-162}\text{Dy}$, $^{164-168}\text{Er}$, $^{168-176}\text{Yb}$, $^{170-178}\text{Hf}$ and the actinides $^{228-232}\text{Th}$, $^{234-238}\text{U}$. The isotopes chosen are all well deformed rotors with energy ratio E_4/E_2 close to 3.3. We have excluded those exhibiting backbending as their proper description requires inclusion of two-quasiparticle states in the model space. While we mainly focus on the description of high-spin states, which has not been done before, we also consider a selected set of low-lying bands. This is important in properly constraining the model parameters so that the results obtained are valid in a broader sense and not just for a small subset of observables. The sdg-IBM parameters used in the fits are listed in table 1. Each parameter is particularly sensitive to a certain set of observables which simplifies the fitting process. For example, q is determined from interband E2 transitions, η_d from the spin dependence of MOI and E_γ (cf. figs. 5-6), η_g from E_{3+} , E_{4+} , ζ_4 from MOI variation (cf. fig. 8), and finally κ_2 from the overall energy scale of the spectrum. The parameters are either constant in a given isotope chain or change smoothly in accordance with the vibration-rotation shape transition, e.g., η_d decreases with increasing N as the nuclei considered become more

rotational.

The representative observables chosen to describe the low-lying band structure are the band excitation energies E_β , E_γ , E_{3+} , E_{4+} (table 2), the interband E2 m.e. for $2_{\beta,\gamma} \rightarrow 0_g$ transitions (table 3), and the E4 m.e. for $4_{\gamma,3+,4+} \rightarrow 0_g$ transitions (table 4). Note that the E4 m.e. are normalized with the ground transition, so that an effective E4 charge is not needed in table 1. With a few exceptions to be discussed below, the general trends of the β and γ band systematics are well reproduced by the calculations. The sudden fluctuations seen in some of the band-head energies (table 2) can be accommodated by a careful tuning of the parameters. Since our aim here is to delineate the systematic features of high-spin states, rather than to obtain refined fits to individual nuclei, we have not attempted such an improvement. Description of interband E2 transitions is one of the strong points of the IBM, and as can be seen from table 3, they are very well reproduced using almost constant q values. The parameters in the E4 operator are determined from the conditions (6), hence the E4 m.e. ratios presented in table 4 are parameter free predictions of the model. Again the overall agreement with the data is reasonable which gives confidence on the choice of the E4 operator. The quality of agreement obtained in tables 2-4 indicates that the limited set of sdg-IBM parameters (table 1) can describe the basic features of the low-lying bands in deformed nuclei.

In the study of high-spin states, we include the level energies for the ground and γ bands, and the yrast E2 transitions for each set of isotopes (figs. 10-27). We first comment on their general features. In all cases, the MOI strongly deviates from the rigid rotor behaviour which would be represented by a horizontal line in the figures. Further, this deviation is not linear but curves up with increasing spin underscoring the importance of the cubic term in eq. (41). Note that because of the ample data available, these features are most clear in the ground bands and to a lesser extent in the γ bands. The yrast E2 m.e., on the other hand, follow closely the rigid rotor values with no sign of a boson cutoff effect. As emphasized in sect. 5, these properties can be explained in the IBM by including the d-boson energy in the Hamiltonian. Below we comment on the specific features of each isotope chain.

1) $^{158-162}\text{Dy}$ (figs. 10-12): Among the deformed nuclei considered in this work,

^{158}Dy , together with ^{170}Hf , exhibit the largest changes in MOI. These nuclei have the lowest boson numbers among the rare-earth set and are clearly influenced by the vibration-rotation phase transition as indicated by the larger η_d values used. The ground band energies (fig. 10) are well described with relative errors of about 1-2%. The trend in γ band energies (fig. 11) is similarly reproduced (note the different scales in figs. 10 and 11). The slight overprediction of energies here can be improved by fine tuning the hexadecapole interaction (cf. fig. 8b). The yrast E2 m.e. have been a sore point in applications of the sd-IBM to high-spin states due to boson cutoff. For example, in ^{158}Dy , the sd-IBM would predict band termination at $L = 26$ which is not seen in the data (fig. 12a). This problem has been resolved in the present sdg-IBM calculations which account for the yrast E2 data very well (fig. 12). A side remark for ^{162}Dy is that the band excitation energies in this nucleus do not follow the trend of $^{158-160}\text{Dy}$ (table 2), hence it requires individual attention for a better description.

2) $^{164-168}\text{Er}$ (figs. 13-15): The Er isotopes, and in particular ^{168}Er , are the exceptional cases mentioned above for which a consistent description of the data could not be obtained with our limited set of parameters. While the spin dependence of the ground and γ band MOI (figs. 13-14) and the E2 m.e. (fig. 15, table 3) are well described, the band excitation energies are overpredicted (table 2) and the E4 m.e. are rather poor (table 4). The problem stems from the fact that, among all the deformed nuclei considered in this study, the Er isotopes have the lowest lying γ bands and the most rigid MOI. As seen from figs. 6-7, these two quantities are correlated in the present parametrization, so that a lower γ band obliges a softer MOI (cf. figs. 6-7). Thus a proper description of the Er isotopes requires extension of the Hamiltonian (4) and/or relaxation of the constraints on the quadrupole and hexadecapole parameters. For example, in a detailed study of ^{168}Er in the sdg-IBM, 14 parameters were employed [5]. Here we will be content with exposing the exceptional nature of the Er isotopes and leave their detailed investigation for future work.

3) $^{168-176}\text{Yb}$ (figs. 16-18): The Yb isotopes are uniformly well described and require little comment. In contrast to Er, the γ band energies in the Yb isotopes are

higher which are well correlated with their relatively stiff MOI (fig. 16). One point worthwhile to make is that the γ band MOI is higher than that of the ground band which could not be explained without the hexadecapole interaction.

4) $^{170-178}\text{Hf}$ (figs. 19-21): In the Hf isotopes, the γ band comes down but the MOI are softer, and hence the correlation between the two quantities is preserved (fig. 19). The staggering observed in the γ bands (fig. 20) requires inclusion of band mixing effects for a better description. Otherwise the data are well reproduced by the calculations.

5) $^{228-232}\text{Th}$ (figs. 22-24): Although boson numbers are lower in the actinide nuclei considered here, they exhibit characteristics of well deformed nuclei. The MOI in actinides are typically twice as large as those in rare-earths requiring smaller κ_2 values. The high-spin data are scarce in $^{228-230}\text{Th}$ but in ^{232}Th , where data up to spin $L = 30$ are available, an excellent description is obtained. One interesting feature of the γ band MOI in ^{232}Th is that it is larger and stiffer compared to the ground band. Both of these features require a large dose of hexadecapole interaction for explanation as remarked in the systematic studies.

6) $^{234-238}\text{U}$ (figs. 25-27): The most extensive high-spin data are available for the yrast bands in the U isotopes which are well described by the present calculations. The yrast E2 m.e. in the U isotopes (and ^{232}Th) were measured to check the boson cutoff predictions of the sd-IBM, i.e. E2 m.e. vanish at $L = 2N$. As seen in fig. 27, the E2 data show no sign of falloff which provides one of the strongest motivations for inclusion of g bosons. At the highest spins, the sdg-IBM calculations appear to underpredict the E2 measurements. We emphasize that this is not due to any boson cutoff effect but rather due to deviation of the data from the rigid rotor values. That is, most models would have difficulty in explaining these E2 transition m.e. which are larger than the rigid rotor values (see, for example, [43]).

7 Summary and conclusions

In this paper, we have presented a systematic description of high-spin states in deformed nuclei within the framework of the sdg-IBM. Such a description has been long overdue but has not been performed earlier due to the technical difficulties in

diagonalizing the sdg-IBM Hamiltonians. We have shown that the $1/N$ expansion formalism, extended to higher orders using computer algebra, provides a viable alternative for this purpose. Systematic studies of the model parameters using the $1/N$ expansion formulas have indicated that some of the long standing problems associated with the description of MOI and E2 transitions in the IBM can be resolved by including the d-boson energy in the Hamiltonian. The hexadecapole interaction has a minor effect on the ground band but could play a decisive role on γ and other excited band properties. This feature of the hexadecapole interaction does not appear to be well appreciated in literature. For example, the recently observed staggering effect in some superdeformed bands have been attributed to the hexadecapole degrees of freedom. If this is true, then it would have profound effects on the neighbouring non-yrast bands.

The application of the sdg-IBM Hamiltonian consisting of single-boson energies, and quadrupole and hexadecapole interactions (with constrained parameters) resulted in a mostly uniform and successful description of both low-lying band structures and high-spin states across the rare-earth and actinide regions. In the past, many experimental results on high-spin states were compared with the sd-IBM calculations with negative connotations. This was presumably due to the lack of the sdg-IBM calculations. We hope that the extensive sdg-IBM results presented in this work will help remedying this situation.

Our limited parametrization did not work as well in the case of the Er isotopes which appear to have rather exceptional properties as exposed in this systematic study. We are planning to carry out a more detailed study of the Er isotopes without imposing the constraints used in this paper. It is amusing to note that ^{168}Er , which has been used as a benchmark case in tests of phenomenological models, may turn out to be the most exceptional of all deformed nuclei.

Appendix A

In cases where the higher order $1/N$ expressions are too complicated to be useful, numerical evaluation of the $1/N$ expansion formulas may provide a more practical

alternative. The reduced normalization integral (12) can be written explicitly as

$$F(N, L) = \int_0^1 dz P_L(z) \left[\sum_i x_i^2 P_i(z) \right]^N, \quad (43)$$

where $P_L(z)$ is a Legendre polynomial. In the sd-IBM, the term in brackets is given by $(c_0 + c_2 z^2)^N$, and in the sdg-IBM by $(c_0 + c_2 z^2 + c_4 z^4)^N$ where c_i are constants involving the mean fields. By expanding the binomial or trinomial, and using

$$\int_0^1 dz P_L(z) z^n = \frac{n!}{(n-L)!!(n+L+1)!!}, \quad (44)$$

eq. (43) can be integrated term by term. This has the advantage that $F(N, L)$ and hence $\langle H \rangle$ are calculated exactly to all orders in $1/N$. The variational problem is solved numerically using the exact m.e. (11,17) and the simplex method. The numerical evaluation can be used, for example, to check the convergence of $1/N$ expressions and determine whether a calculation to a particular order is accurate enough.

References

- [1] F. Iachello and A. Arima, The interacting boson model (Cambridge University Press, Cambridge, 1987).
- [2] A. Bohr and B.R. Mottelson, Phys. Scripta 22 (1980) 468.
- [3] A. Bohr and B.R. Mottelson, Phys. Scripta 25 (1982) 28.
- [4] N. Yoshinaga, A. Arima and T. Otsuka, Phys. Lett. B 143 (1984) 5.
- [5] N. Yoshinaga, Y. Akiyama and A. Arima, Phys. Rev. C 38 (1988) 419.
- [6] R.F. Casten and D.D. Warner, Rev. Mod. Phys. 60 (1988) 389.
- [7] Y.D. Devi and V.K.B. Kota, Pramana-J. Phys. 39 (1992) 413.
- [8] K. Heyde, in: Algebraic approaches to nuclear structure, ed. R.F. Casten (Hardwood Academic Publishers, Switzerland, 1993) p. 323.
- [9] P. Van Isacker, K. Heyde, M. Waroquier and G. Wenes, Nucl. Phys. A 380 (1982) 383.
- [10] V.S. Lac and S. Kuyucak, Nucl. Phys. A 539 (1992) 418.
- [11] S. Kuyucak and I. Morrison, Ann. Phys. (N.Y.) 181 (1988) 79; *ibid*, 195 (1989) 126.
- [12] S. Kuyucak, I. Morrison and T. Sebe, Phys. Rev. C 43 (1991) 1187.
- [13] S. Kuyucak and S.C. Li, Phys. Lett. B 349 (1995) 253.
- [14] S. Kuyucak and S.C. Li, Phys. Lett. B 354 (1995) 189.
- [15] T. Otsuka, A. Arima and F. Iachello, Nucl. Phys. A 309 (1978) 1.
- [16] I. Morrison, Computer code SDGBOSON (University of Melbourne, 1986).
- [17] S.C. Li, Computer code SDGBOSON for the supercomputer (Australian National University, 1993).

- [18] J.N. Ginocchio and M.W. Kirson, Nucl. Phys. A 350 (1980) 31.
- [19] S. Wolfram, Mathematica (Addison-Wesley, Redwood City, 1991).
- [20] S. Kuyucak and K. Unnikrishnan, J. Phys. A 28 (1995) 2101.
- [21] S. Kuyucak and I. Morrison, Phys. Rev. C 38 (1988) 2482.
- [22] S. Kuyucak and S.C. Li, Computer code IBM-1/N (Australian National University, 1995).
- [23] S. Kuyucak and I. Morrison, Phys. Rev. C 41 (1990) 1803.
- [24] A. Bohr and B.R. Mottelson, Nuclear structure, Vol. 2 (Benjamin, Reading, 1975).
- [25] N. Yoshida, H. Sagawa, T. Otsuka and A. Arima, Phys. Lett. B 256 (1991) 129.
- [26] P.O. Lipas, P. Toivonen and D.D. Warner, Phys. Lett. B 155 (1985) 295.
- [27] M.A. Lee, Nucl. Data Sheets [A158] 56 (1989) 199.
- [28] M.A. Lee and R.L. Bunting, Nucl. Data Sheets [A160] 46 (1985) 187.
- [29] R.G. Helmer, Nucl. Data Sheets [A162] 44 (1985) 659.
- [30] E.N. Shurshikov and N.V. Timofeeva, Nucl. Data Sheets [A164] 65 (1992) 365.
- [31] E.N. Shurshikov and N.V. Timofeeva, Nucl. Data Sheets [A166] 67 (1992) 45.
- [32] V.S. Shirley, Nucl. Data Sheets [A168] 71 (1994) 261.
- [33] C. Zhou, Nucl. Data Sheets [A170] 50 (1987) 351.
- [34] G. Wang, Nucl. Data Sheets [A172] 51 (1987) 557.
- [35] E. Browne, Nucl. Data Sheets [A174] 62 (1991) 1.
- [36] E. Browne, Nucl. Data Sheets [A176] 60 (1990) 227.
- [37] E. Browne, Nucl. Data Sheets [A178] 72 (1994) 221.

- [38] M.J. Martin, Nucl. Data Sheets [A228] 49 (1986) 83.
- [39] Y.A. Akovali, Nucl. Data Sheets [A230] 69 (1993) 155.
- [40] M.R. Schmorak, Nucl. Data Sheets [A232, A236] 63 (1991) 139.
- [41] Y.A. Akovali, Nucl. Data Sheets [A234] 71 (1994) 181.
- [42] E.N. Shurshikov, Nucl. Data Sheets [A238] 53 (1988) 601.
- [43] D. Troltenier, J.P. Draayer, P.O. Hess and O. Castanos, Nucl. Phys. A 576 (1994) 351.

Figure captions

Fig. 1. Effect of the basis space truncation on a) band excitation energies b) E2 transitions, and c) E4 transitions. The maximum number of g bosons, $n_{g\max}$, allowed in the basis space is increased from 1 to the maximum of $N = 10$. Parameters of the sdg-IBM Hamiltonian are given in the text.

Fig. 2. Effect of the basis space truncation on the ground band a) excitation energies, and b) E2 transitions.

Fig. 3. Comparison of the ground band energies obtained from the $1/N$ expansion with the exact diagonalization results (circles). In (a) different lines refer to the second layer calculation (dotted line), the third layer (dashed line), and the numerical one to all orders (solid line). In (b) the lines correspond to the third layer results obtained with $\eta_d = 1.5$, $\eta_g = 4.5$ (top), $\eta_d = 0$, $\eta_g = 4.5$ (middle), $\eta_d = 0$, $\eta_g = 0$ (bottom).

Fig. 4. (a) Comparison of the γ band energies obtained from the $1/N$ expansion with the exact diagonalization results (circles). (b) Comparison of the yrast E2 transition m.e. obtained from eq. (42) (dashed line) with the exact diagonalization results (circles). The solid line shows the numerical evaluation of the m.e. to all orders.

Fig. 5. The effect of (a) the d-boson energy and (b) the hexadecapole interaction on the ratio $E_\gamma/N\lambda_{g1}$ which relates the energy scales of the γ and ground bands.

Fig. 6. Same as fig. 5 but for the ratio $N^2\lambda_{g2}/\lambda_{g1}$ which measures the deviation of the ground band moment of inertia from the rigid rotor behaviour.

Fig. 7. Same as fig. 6 but for the γ band.

Fig. 8. Same as fig. 5 but for the ratio $\lambda_{\gamma1}/\lambda_{g1}$ which compares the moment of inertia of the ground and γ bands.

Fig. 9. Same as fig. 5 but for the ratio m_2/m_1 which measures the boson cutoff

effect.

Fig. 10. Comparison of the experimental (circles) and calculated (solid lines) ground band energies $E_{gL}/L(L+1)$ (in keV) in $^{156-160}\text{Dy}$. The data are from [27, 28, 29].

Fig. 11. Same as fig. 10 but for the γ bands.

Fig. 12. Comparison of the experimental (circles) and calculated (solid lines) yrast $E2$ transitions in $^{156-160}\text{Dy}$. The data are from [27, 28, 29].

Fig. 13. Same as fig. 10 but in $^{164-168}\text{Er}$. The data are from [30, 31, 32].

Fig. 14. Same as fig. 11 but in $^{164-168}\text{Er}$. The data are from [30, 31, 32].

Fig. 15. Same as fig. 12 but in $^{164-168}\text{Er}$. The data are from [30, 31, 32].

Fig. 16. Same as fig. 10 but in $^{168-176}\text{Yb}$. The data are from [32, 33, 34, 35, 36].

Fig. 17. Same as fig. 11 but in $^{168-176}\text{Yb}$. The data are from [32, 33, 34, 35, 36].

Fig. 18. Same as fig. 12 but in $^{168-176}\text{Yb}$. The data are from [32, 33, 34, 35, 36].

Fig. 19. Same as fig. 10 but in $^{170-178}\text{Hf}$. The data are from [33, 34, 35, 36, 37].

Fig. 20. Same as fig. 11 but in $^{170-178}\text{Hf}$. The data are from [33, 34, 35, 36, 37].

Fig. 21. Same as fig. 12 but in $^{170-178}\text{Hf}$. The data are from [33, 34, 35, 36, 37].

Fig. 22. Same as fig. 10 but in $^{228-232}\text{Th}$. The data are from [38, 39, 40].

Fig. 23. Same as fig. 11 but in $^{228-232}\text{Th}$. The data are from [38, 39, 40].

Fig. 24. Same as fig. 12 but in $^{228-232}\text{Th}$. The data are from [38, 39, 40].

Fig. 25. Same as fig. 10 but in $^{234-238}\text{U}$. The data are from [41, 40, 42].

Fig. 26. Same as fig. 11 but in $^{234-238}\text{U}$. The data are from [41, 40, 42].

Fig. 27. Same as fig. 12 but in $^{234-238}\text{U}$. The data are from [41, 40, 42].

Table 1: Parameters used in the sdg-IBM calculations. κ_2 is in keV and e_2 in eb.

Nucleus	N	κ_2	ζ_4	q	η_d	η_g	e_2
¹⁵⁸ Dy	13	19.8	0.30	0.50	1.90	5.0	0.13
¹⁶⁰ Dy	14	19.9	0.30	0.50	1.77	4.6	0.13
¹⁶² Dy	15	19.3	0.35	0.50	1.60	4.5	0.13
¹⁶⁴ Er	14	22.0	0.40	0.50	1.50	5.0	0.14
¹⁶⁶ Er	15	21.2	0.40	0.50	1.42	4.7	0.13
¹⁶⁸ Er	16	23.1	0.40	0.50	1.22	4.6	0.13
¹⁶⁸ Yb	14	19.7	0.35	0.50	1.68	5.0	0.14
¹⁷⁰ Yb	15	20.6	0.35	0.50	1.63	4.7	0.13
¹⁷² Yb	16	20.5	0.25	0.60	1.78	5.0	0.13
¹⁷⁴ Yb	17	22.2	0.25	0.60	1.66	4.1	0.12
¹⁷⁶ Yb	16	20.1	0.35	0.60	1.83	5.3	0.12
¹⁷⁰ Hf	13	19.1	0.10	0.50	2.04	4.9	0.14
¹⁷² Hf	14	19.5	0.10	0.50	1.99	4.5	0.13
¹⁷⁴ Hf	15	20.7	0.10	0.50	1.91	4.1	0.13
¹⁷⁶ Hf	16	22.0	0.10	0.50	1.80	3.5	0.13
¹⁷⁸ Hf	15	21.8	0.10	0.50	1.83	4.1	0.13
²²⁸ Th	10	19.7	0.30	0.68	1.60	3.0	0.20
²³⁰ Th	11	15.5	0.40	0.68	1.59	4.4	0.20
²³² Th	12	14.6	0.40	0.68	1.58	4.4	0.20
²³⁴ U	13	14.9	0.20	0.70	1.62	3.5	0.18
²³⁶ U	14	16.4	0.20	0.70	1.57	3.2	0.17
²³⁸ U	15	17.7	0.20	0.70	1.56	2.8	0.17

Table 2: Comparison of the β , γ , and $K = 3^+, 4^+$ (single-phonon) band energies (in keV) with the sdg-IBM calculations in the rare-earth and actinide regions. The data are from Nucl. Data Sheets.

Nucleus	E_β		E_γ		E_{3^+}		E_{4^+}	
	Cal.	Exp.	Cal.	Exp.	Cal.	Exp.	Cal.	Exp.
^{158}Dy	916	991	965	946	1461	-	1935	1895
^{160}Dy	1204	1275	998	966	1512	-	2085	-
^{162}Dy	1284	1205	1048	888	1617	-	2201	1536
^{164}Er	1233	1246	1145	860	1634	1702	2003	-
^{166}Er	1275	1460	1138	786	1704	-	2376	-
^{168}Er	1590	1217	1345	821	1892	1654	2483	2238
^{168}Yb	1085	1156	1028	984	1470	1452	1706	-
^{170}Yb	1218	1069	1138	1145	1501	-	1857	1408
^{172}Yb	1345	1043	1385	1466	1799	1663	2403	2073
^{174}Yb	1589	1487	1554	1634	1810	-	2545	-
^{176}Yb	1329	1779	1324	1261	1685	-	2407	-
^{170}Hf	868	880	985	961	1482	-	1528	-
^{172}Hf	982	871	1016	1075	1530	-	1604	-
^{174}Hf	1108	827	1137	1227	1547	1303	1669	-
^{176}Hf	1298	1150	1278	1341	1671	1578	1840	-
^{178}Hf	1231	1199	1202	1175	1710	1728	1809	1848
^{228}Th	791	832	846	969	1082	-	1458	-
^{230}Th	734	635	792	781	1004	-	1519	-
^{232}Th	769	730	813	785	1041	-	1574	-
^{234}U	825	809	871	927	1253	1496	1566	1723
^{236}U	986	919	1002	958	1325	-	1799	-
^{238}U	1080	993	1086	1060	1328	1059	1965	-

Table 3: Comparison of the interband E2 transitions (in *eb*) with the sdg-IBM calculations in the rare-earth and actinide regions. The data are from Nucl. Data Sheets.

Nucleus	$\langle 2_\beta T(E2) 0_g \rangle$		$\langle 2_\gamma T(E2) 0_g \rangle$	
	Cal.	Exp.	Cal.	Exp.
¹⁵⁸ Dy	0.21	0.23 ± 0.02	0.41	0.39 ± 0.04
¹⁶⁰ Dy	0.19	-	0.34	0.27 ± 0.04
¹⁶² Dy	0.17	-	0.35	0.35 ± 0.02
¹⁶⁴ Er	0.17	-	0.34	0.37 ± 0.02
¹⁶⁶ Er	0.16	-	0.41	0.39 ± 0.02
¹⁶⁸ Er	0.15	<0.03	0.34	0.36 ± 0.01
¹⁶⁸ Yb	0.20	0.22 ± 0.01	0.35	0.36 ± 0.04
¹⁷⁰ Yb	0.19	0.17 ± 0.02	0.32	0.28 ± 0.03
¹⁷² Yb	0.16	0.09 ± 0.01	0.26	0.21 ± 0.03
¹⁷⁴ Yb	0.14	-	0.24	0.22 ± 0.03
¹⁷⁶ Yb	0.14	-	0.27	0.23 ± 0.03
¹⁷⁰ Hf	0.23	-	0.36	-
¹⁷² Hf	0.19	-	0.31	-
¹⁷⁴ Hf	0.20	-	0.33	0.37 ± 0.04
¹⁷⁶ Hf	0.18	0.21 ± 0.02	0.34	0.35 ± 0.01
¹⁷⁸ Hf	0.21	-	0.32	0.34 ± 0.02
²²⁸ Th	0.18	-	0.28	-
²³⁰ Th	0.20	0.21 ± 0.05	0.34	0.35 ± 0.06
²³² Th	0.23	0.31 ± 0.07	0.38	0.36 ± 0.04
²³⁴ U	0.21	<0.24	0.32	0.35 ± 0.04
²³⁶ U	0.20	-	0.30	-
²³⁸ U	0.19	0.23 ± 0.03	0.31	0.36 ± 0.04

Table 4: Comparison of the interband $E4$ transitions, normalized to inband ones, with the sdg-IBM calculations in the rare-earth and actinide regions. The data are from Nucl. Data Sheets.

Nucleus	$\frac{\langle 4_{\gamma} T(E4) 0_g \rangle}{\langle 4_g T(E4) 0_g \rangle}$		$\frac{\langle 4_{3+} T(E4) 0_g \rangle}{\langle 4_g T(E4) 0_g \rangle}$		$\frac{\langle 4_{4+} T(E4) 0_g \rangle}{\langle 4_g T(E4) 0_g \rangle}$	
	Cal.	Exp.	Cal.	Exp.	Cal.	Exp.
^{158}Dy	0.49	0.56 ± 0.44	1.14	-	0.32	-
^{160}Dy	0.50	0.63 ± 0.24	1.08	-	0.27	-
^{162}Dy	0.37	-	1.06	-	0.28	-
^{164}Er	0.44	-	1.05	-	0.23	-
^{166}Er	0.53	-	1.00	-	0.18	-
^{168}Er	0.46	1.32 ± 0.72	0.95	0.60 ± 0.33	0.15	0.25 ± 0.15
^{168}Yb	0.50	-	0.95	-	0.20	-
^{170}Yb	0.60	-	0.89	-	0.14	-
^{172}Yb	0.53	0.20 ± 0.19	0.82	0.69 ± 0.57	0.09	-
^{172}Yb	0.36	0.20 ± 0.19	0.78	0.69 ± 0.57	0.16	-
^{174}Yb	0.44	-	0.73	0.54 ± 0.29	0.11	-
^{176}Yb	0.37	-	0.81	-	0.17	-
^{170}Hf	0.44	-	1.17	-	0.43	-
^{172}Hf	0.45	-	1.12	-	0.38	-
^{174}Hf	0.50	-	1.04	-	0.28	-
^{176}Hf	0.44	-	1.01	-	0.28	-
^{178}Hf	0.43	-	1.08	-	0.34	-
^{228}Th	0.55	-	0.62	-	0.02	-
^{230}Th	0.44	-	0.71	-	0.07	-
^{232}Th	0.48	-	0.68	-	0.05	-
^{234}U	0.48	-	0.61	-	0.02	-
^{238}U	0.50	-	0.58	-	0.02	-